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Shock-Like Solutions of the Electrostatic Vlasov Equation*

by

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ABSTRACT

It is shown how to construct shock-like time independent solutions of the electrostatic Vlasov and Poisson Equations in one dimension. The positive ions are assumed to be at zero temperature. The electrostatic potential is assumed to increase monotonically through the shock from zero to a constant value. The most important feature of the solution is a population of trapped electrons in the shocked plasma. In contrast to time-independent solutions based upon fluid equations, there is no upper limit on the amplitude of the shock.

I. INTRODUCTION

The subject of collisionless shocks occupies an extensive literature in plasma physics. A review and bibliography are given by Sagdeev [1966]. Many of the treatments have been concerned with situations in which the ratio of magnetic to mechanical stress is sufficiently large that the magnetic field may be taken to be basic to the shock structure. Here, we shall be concerned with a different limit, the purely electrostatic one, which involves no magnetic field or net electrical currents at all.

Existing electrostatic shock theories are divided by Sagdeev into two types, "laminar" and "turbulent". In the "laminar" theories, one starts with a time-independent set of solutions to whatever dynamical equations are being used. Characteristically one obtains in this way a shock profile which oscillates indefinitely with space. The oscillations must be damped if one is to obtain a uniform state at infinity. Typically, this is done by the addition of a small dissipation (e.g., collisions) on a somewhat ad hoc basis.

"Turbulent" shock theories rely on a high degree of random oscillation within the shock front to provide a dissipative mechanism, thereby at one stroke providing a very appealing physical picture of the shock, but also putting most of the details of the shock's structure outside the boundary of those phenomena which are likely to be analytically manageable in the foreseeable future. (This remark is predicated on the belief that a satisfactory analytical theory of plasma turbulence will be at least as hard to come by as the corresponding theory for hydrodynamic turbulence has proved to be.)

The most detailed and convincing turbulent shock theory to date is probably that of Tidman [1967]. Proceeding from an assumed Mott-Smith distribution for the incoming and outgoing plasmas, the ion-wave instability is regarded as the origin of a spectrum of electrostatic waves which effects the transition from the upstream to the downstream state of the plasma. A very complicated set of dynamical equations results, however, and except for the leading edge of the shock, one can only conjecture properties of their solutions.

The present paper attempts nothing so ambitious as a definitive theory of a collisionless shock transition, or even to decide in favor of the laminar or turbulent concept. We simply show here how laminar solutions, which lead to different upstream and downstream spatially uniform states, can be constructed entirely within the time-independent Vlasov framework, without the introduction of any dissipative mechanism *per se*. We find that the presence of a population of *trapped electrons* (electrons with negative total energy, relative to the zero of electrostatic potential) will permit the construction of a wide class of shock-like solutions to the time-independent Vlasov-Poisson system.

It has been realized for several years [Bernstein, Greene, and Kruskal, 1957; Harris, 1957] that trapped particle distributions provided considerable freedom to construct different time-independent but spatially varying solutions to the Vlasov-Poisson system. These results, however, have been to a great extent on a formal level, and it has been feared that the required trapped particle distributions may be negative or otherwise physically unrealistic. We show that, at least in the present case, such fear is groundless.

We use, for the gross properties of our shock, experimental indications provided by the measurements of Andersen, D'Angelo, Michelsen, and Nielsen [1967] (an earlier and somewhat less clear-cut experiment in spherical geometry was due to Koopman and Tidman [1967]). Both this experiment and calculations on nonlinear ion acoustic waves [Montgomery, 1967] indicate that the shock is most likely to be present when the plasma electrons are very hot relative to the ions. We set the ion temperature equal to zero. We model the field as that provided by a monotonically increasing electrostatic potential (see Figure 1a). The downstream ($x \to -\infty$) plasma is assumed to be at the higher density.

For present purposes, our problem will be taken to be determining what sorts of time-independent solutions to the electrostatic collisionless plasma equations exist, subject to the following restrictions:

(1) The electric field, $E(x) = -\varphi'(x)$, and the particle distribution functions depend only on one spatial coordinate (x, say).

(2) The electrostatic potential, $\varphi(x)$, is non-negative, $\to 0$ as $x \to +\infty$, and $\to \varphi_0 > 0$ monotonically as $x \to -\infty$.

(3) The positive ions are cold (have no spread in velocity) and the electrons are at a finite temperature.

(4) The number densities of electrons and (singly-charged) positive ions are equal at $x = +\infty$ and at $x = -\infty$, but are larger at $x = -\infty$ (downstream from the shock) than at $x = +\infty$.

(5) The shock amplitude φ_0 is not strong enough to turn around any of the ions, but a population of "trapped" (negative total energy) electrons may exist.

Three separate facets of the problem must be considered, the ion dynamics, the electron dynamics, and Poisson's equation. These are taken up in the following sections. The general properties of the solutions are summarized in Section V.

II. ION DYNAMICS

We assume that the electrons and ions both pour in from $x = \infty$ with number density n_o and particle current $- n_o V_o$. The ion velocity $u_i(x)$ and number density $n_i(x)$ satisfy

$$n_i(x)\, u_i(x) = \text{const.} = - n_o V_o \,, \tag{1}$$

by the ion equation of continuity. Since there is no spread in ion velocities, the velocity u_i may be related to V_o and $\varphi(x)$ by conservation of energy

$$\tfrac{1}{2} m_i\, u_i^{\,2}(x) + e\,\varphi(x) = \tfrac{1}{2} m_i\, V_o^{\,2} \,, \tag{2}$$

where m_i is the ion mass and $e = |e|$ is the charge.

We assume $(m_i V_o^2/2) > e\,\varphi_o$, so the ions pass on across the shock to the left and have a velocity $-V_o(1-2\,e\,\varphi_o/m_i V_o^2)^{\frac{1}{2}}$ at $x = -\infty$.

The charge density of the ions is, from Equations (1) and (2),

$$e\, n_i(x) = e\, n_o (1 - 2\,e\,\varphi(x)/m_i\, V_o^{\,2})^{-\frac{1}{2}} \,. \tag{3}$$

The schematic form of the potential $\varphi(x)$ is shown in Figure 1a, and the corresponding ion orbit in the ion x, v phase space is shown in Figure 1b.

III. ELECTRON DYNAMICS

The electron thermal energy will be treated as of the same order as $e\,\varphi_o$, so in general both "free" and "trapped" electrons will be involved. By free and trapped, we mean electrons for which the total energy

$$\Sigma = \tfrac{1}{2} m_e v^2 - e\,\varphi(x) \tag{4}$$

is positive or negative, respectively. For $\varphi(x)$ of the form chosen, positive energy electrons will never reverse their sign of v, negative energy electrons will reverse it exactly once. The separatrix between the trapped and untrapped parts of the electron phase plane (shown in Figure 1c)is the pair of curves $v = \pm\sqrt{2e\varphi(x)/m_e}$.

The electron distribution function f_e will hereafter be divided into "free" and "trapped" parts:

$$f_e(x, v) = f_{ef}(x, v) + f_{et}(x, v) ,$$

where $f_{ef} = 0$ if $\Sigma < 0$ and $f_{et} = 0$ if $\Sigma > 0$. Both f_{ef} and f_{et} will be functions of Σ only [Bernstein, Greene, and Kruskal, 1957]. Only f_{ef} can be determined by giving its value at $x = +\infty$, ahead of the shock. It appears to be one of the irreducible ambiguities in the collisionless shock problem [see, e.g., Morawetz, 1962] that there remains a degree of arbitrariness in the trapped particle population. We shall see later, however, that this arbitrary trapped particle population is what permits shock-like solutions to exist, and that a wide variety of such solutions can exist with restrictions on the trapped particle distribution which are not at all severe.

For f_{ef}, it is natural to assume a Maxwellian at $x = +\infty$:

$$\lim_{x \to \infty} f_{ef}(x, v) = n_o \left(\frac{m_e}{2\pi K T_e} \right)^{\frac{1}{2}} \exp \left\{ \frac{- m_e (v + V_o)^2}{2 K T_e} \right\} . \qquad (5)$$

Expressing v in terms of Σ, this means that at finite x, we will have

$$f_{ef}(x, v) = n_o \left(\frac{m_e}{2\pi K T_e} \right)^{\frac{1}{2}} \begin{cases} \exp \left[\frac{-m_e}{2KT_e} \left(\sqrt{v^2 - \frac{2e\varphi}{m_e}} + V_o \right)^2 \right] \\ \exp \left[\frac{-m_e}{2KT_e} \left(\sqrt{v^2 - \frac{2e\varphi}{m_e}} - V_o \right)^2 \right] \end{cases} \qquad (6)$$

where the upper and lower expressions apply to the regions $v > 0$ and $v < 0$ respectively. Equation (6) reduces to Equation (5) as

$\varphi \xrightarrow[x \to \infty]{} 0$, and is clearly a function only of the constant of the motion Σ.

The free charge density is, for the electrons,

$$- e\, n_{ef}(x) =$$

$$- e\, n_o \left(\frac{m_e}{2\pi K T_e} \right)^{\frac{1}{2}} \int_{+\sqrt{2e\varphi/m_e}}^{+\infty} dv \quad \exp \left\{ \frac{-m_e}{2KT_e} \left(\sqrt{v^2 - \frac{2e\varphi}{m_e}} + V_o \right)^2 \right\}$$

$$- e\, n_o \left(\frac{m_e}{2\pi K T_e} \right)^{\frac{1}{2}} \int_{-\infty}^{-\sqrt{2e\varphi/m_e}} dv \quad \exp \left\{ \frac{-m_e}{2KT_e} \left(\sqrt{v^2 - \frac{2e\varphi}{m_e}} - V_o \right)^2 \right\} \tag{7}$$

The v-integrations in Equation (7) are complicated but are simplified by the following observation. We anticipate that V_o will be of the order of the ion acoustic speed, $\sqrt{KT_e/m_i}$. In fact, we set

$$V_o^{\,2} = M^2\, KT_e/m_i \quad , \tag{8}$$

where we expect the "Mach number" M to be somewhat greater than unity, but not vastly greater. The integrals in Equation (7) come largely from regions of v where $m_e(\sqrt{v^2-2e\varphi/m_e} \pm V_o)^2$ is of the order of KT_e. Thus most of the contribution comes from a region

$$\sqrt{v^2 - 2e\varphi/m_e} \cong O\left(\sqrt{\frac{KT_e}{m_e}} >> V_o\right).$$

We can then ignore the $\pm V_o$ in Equation (7) to a first approximation, and write

$$- e\, n_{ef}(x) = - 2e\, n_o \left(\frac{m_e}{2\pi KT_e}\right)^{\frac{1}{2}} \exp\left(\frac{e\varphi}{KT_e}\right) \int_{\sqrt{2e\varphi/m_e}}^{+\infty} dv \quad \exp\left(\frac{- m_e v^2}{2KT_e}\right) \tag{9}$$

up to terms of $O(M\sqrt{\frac{m_e}{m_i}})$.

The distribution functions of the free and trapped electrons are shown schematically in Figure 2a. We purposely leave $f_{et}(\Sigma)$ unspecified at this point. The contribution of $f_{et}(\Sigma)$ to the electron charge density is given by $- e\, n_{et}(x)$, where

$$n_{et}(x) \equiv n_{et}(e\varphi) = \int_{-e\varphi}^{0} \frac{d\Sigma\, f_{et}(\Sigma)}{\sqrt{2m_e(\Sigma+e\varphi)}} \tag{10}$$

It is clear that since $f_{et}(\Sigma) \geq 0$, n_{et} is always a non-negative non-decreasing function of $e\varphi$, as shown in Figure 2b, and that $n_{et}(0) = 0$. In the special case when there are no trapped electrons, and only then, n_{et} will vanish for all φ.

It is clear that, in fact, n_i, n_{ef} and n_{et} are all functions of the scalar potential φ alone, once the distribution functions are given.

IV. POISSON'S EQUATION

The remaining problem is the satisfaction of Poisson's Equation, which, collecting the results of Equations (3), (9), and (10), becomes

$$\varphi''(x) = -4\pi e n_o (1 - 2e\varphi/m_i V_o^2)^{-\frac{1}{2}} + 4\pi e n_{ef}(e\varphi) + 4\pi e n_{et}(e\varphi) \; . \tag{11}$$

It is convenient to reduce Equation (10) to dimensionless form by defining

$$\Psi \equiv \frac{e\varphi}{KT_e}$$

$$\xi^2 \equiv \frac{4\pi n_o e^2}{KT_e} x^2 \equiv \frac{x^2}{(\text{Debye length})^2} \; ,$$

$$\frac{n_{et}(e\varphi)}{n_o} \equiv \alpha(\Psi) \geq 0 \; ,$$

and

$$\frac{2KT_e}{m_i V_o^2} \equiv \frac{2}{M^2} \equiv \delta^2$$

(we expect $\delta^2 < 2$).

With these definitions, Poisson's equation becomes

$$\frac{d^2\Psi(\xi)}{d\xi^2} = -(1-\delta^2\Psi)^{-\frac{1}{2}} + (1-\mathrm{erf}\sqrt{\Psi})e^{\Psi} + \alpha(\Psi)\ . \tag{12a}$$

It is convenient to rewrite (12a) as

$$\frac{d^2\Psi(\xi)}{d\xi^2} = -\frac{\partial V(\Psi)}{\partial\Psi}\ , \tag{12b}$$

where

$$V(\Psi) \equiv U(\Psi) - A(\Psi)\ , \tag{13a}$$

and

$$U(\Psi) \equiv -\left(2/\delta^2\right) + \left(2/\delta^2\right)(1-\delta^2\Psi)^{\frac{1}{2}} - \int_0^{\Psi} d\theta(1-\mathrm{erf}\sqrt{\theta})\, e^{\theta}\ , \tag{13b}$$

$$A(\Psi) \equiv \int_0^{\Psi} \alpha(\theta)\, d\theta\ . \tag{13c}$$

Note that $U(0) = 0$, $U'(0) = 0$, and $U''(\Psi) = O(\Psi^{-\frac{1}{2}})$ for Ψ small and positive. $A(\Psi)$ is the trapped electron contribution to U, and is undetermined at this point.

Equation (12b) is now formally the equation of motion of a fictitious "particle" of "position" Ψ moving in a "potential" $V(\Psi)$. The dimensionless length ξ plays the role of the "time". The solution to this equivalent mechanics problem is well known, and in what follows we shall use language appropriate to it, bearing in mind the just-described mathematical correspondences with the problem at hand (this device appears to have been introduced by Davis, Lüst, and Schlüter [1958]).

Equation (12) has an "energy" integral:

$$\frac{1}{2}\left(\frac{d\Psi}{d\xi}\right)^2 + V(\Psi) = \text{const.} \equiv \Pi, \text{ say.} \tag{14a}$$

This reduces at once to the quadrature

$$\int \frac{d\Psi}{\sqrt{\Pi - V(\Psi)}} = \pm \int \sqrt{2}\ d\xi\ . \tag{14b}$$

Solutions to Equation (14) in which Ψ remains bounded are clearly only possible if $V(\Psi) = \Pi$ at the upper and lower bounds of Ψ

(in our problem, at $\Psi = 0$ and $\Psi = \Psi_o > 0$); otherwise Ψ will grow without bound in one direction or the other. It is equally clear (see Figure 3a) that $V(\Psi)$ must be less than Π in the interval $0 < \Psi < \Psi_o$. Thus the requirements

$$V(0) = V(\Psi_o) \tag{15a}$$

$$V(0) < V(\Psi), \quad 0 < \Psi < \Psi_o \quad , \tag{15b}$$

follow only from the requirement of boundedness.

As noted by Sagdeev [1966], most of the solutions $\Psi(\xi)$ which result from Equations (14) and (15) are <u>periodic</u> functions of ξ with finite interval of periodicity. Since we are trying to construct solutions in which Ψ goes monotonically from zero to $\Psi_o > 0$ as ξ goes from $+\infty$ to $-\infty$, these are of no use.

The only exception to this statement occurs when the horizontal line $V = \Pi$ intersects $V(\Psi)$ at <u>local</u> <u>maxima</u>. Examination of the behavior of Ψ near these turning points readily reveals that for this situation, the interval of periodicity in ξ becomes <u>infinite</u>. The way to get Ψ to go from zero to $\Psi_o > 0$ monotonically as ξ goes from $+\infty$ to $-\infty$ is to have this infinite periodicity requirement fulfilled at both end points of the motion; i.e., to require in addition to Equation (15), that:

(1) $V(0)$ and $V(\Psi_o)$ are local maxima; and

(2) $\Pi = 0$. (16)

Then if $\Psi(\xi = +\infty)$ is positive and arbitrarily small, Ψ will approach $\Psi_o > 0$ monotonically as $\xi \to -\infty$, and we will have our shock-like solution. This state of affairs is summarized in Figure 3b.

We have now to consider what restrictions are imposed on the trapped electron distribution by the requirement that $V(\Psi)$ have the shape shown in Figure 3. It is not obvious at this point that there even exist values of $f_{et}(\Sigma)$ such that $V(\Psi)$ will have the appropriate form.

To this end, it is useful to consider $U(\Psi)$, which is what $V(\Psi)$ would be if there were no trapped particles. A numerical plot of $U(\Psi)$ for various values of $\delta^2 \leq 2$ is shown in Figure 4. It can be proved with complete rigor that $U(\Psi)$, $U'(\Psi)$, and $U''(\Psi)$ are always positive in the interval $0 < \Psi \leq \delta^{-2}$. This shows first of all that no solutions of the type we are seeking exist for the case of <u>no</u> trapped particles; trapped electrons are essential.

Attention now focuses on $A(\Psi)$, the trapped particle contribution to $V(\Psi)$. $A(\Psi)$ is positive, and must be subtracted off from $U(\Psi)$ in such a way that

(1) $A(\Psi) - U(\Psi) > 0, \quad 0 < \Psi < \Psi_o < \delta^{-2}$,

(2) $A'(\Psi_o) = U'(\Psi_o)$.

We already have $A'(0) = 0$. If $A(\Psi)$ can be found which meets these two requirements, it is clear that $V(\Psi) = U(\Psi) - A(\Psi)$ will in fact have the form shown in Figure 3, and thus lead to the desired solution.

As seen in Figure 5 for a typical $U(\Psi)$, it is geometrically obvious that many such $A(\Psi)$'s (an infinite number, in fact) can be constructed for which the above two conditions are fulfilled. It is important also to note that these $A(\Psi)$'s may be drawn such that $A'(\Psi) > 0$ and $A''(\Psi) > 0$ for $0 < \Psi < \Psi_o$ = the point of intersection. The fact that $A(\Psi)$ has positive curvature will later be shown to lead to the guarantee that $f_{et}(\Sigma)$ be positive semi-definite.

Given any such $A(\Psi)$, it is now a simple matter to find the trapped particle distribution which will support it. Taking $\alpha(\Psi) = A'(\Psi)$, and writing Equation (10) in dimensionless variables,

$$\beta(\Psi) = \int_{-\Psi}^{0} \frac{d\epsilon \, f_{et}(\epsilon)}{\sqrt{\epsilon + \Psi}} \tag{17}$$

where $\epsilon \equiv \Sigma/KT_e$, $\beta(\Psi) \equiv A'(\Psi)\, n_o \sqrt{2m_e/KT_e}$. As Bernstein, Greene, and Kruskal pointed out, Equation (17) is just Abel's equation, if we choose to regard it as an integral equation for the trapped electron distribution f_{et}. Its solution is

$$f_{et}(\epsilon) = \frac{n_o}{\pi}\sqrt{\frac{2m_e}{KT_e}} \int_o^{|\epsilon|} \frac{A''(\theta)\, d\theta}{\sqrt{-\epsilon - \theta}} \qquad (18)$$

for

$$0 > \epsilon = \frac{\Sigma}{KT_e} > - \Psi = \frac{e\varphi}{KT_e} .$$

By virtue of the positive definiteness of $A''(\theta)$ noted above, Equation (18) shows explicity that the trapped electron distribution is non-negative. (The requirement that A'' be positive is still even unnecessarily stringent, and can be relaxed considerably.)

Since the dynamical equations and Poisson's equation have both been solved and the resulting quantities shown to satisfy the various physical requirements, this completes the problem.* An infinite number of solutions exist which will take us monotonically from 0 to φ_o as x goes from $+\infty$ to $-\infty$, and they differ only in the functional form of $\varphi(x)$ that connects the two end points.

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*We should note one last requirement on $A(\Psi)$, which is of little or no consequence. In order that $V(\Psi)$ lead to infinite periodicity at the end points, $V(\Psi)$ must vary as the square of the quantities $|\Psi|$ and $|\Psi_o - \Psi|$, respectively, at the two end points. This implies that near $\Psi = 0$, $A(\Psi)$ must contain a term which $\sim \Psi^{3/2}$ to cancel off the leading term in $U(\Psi)$ which also $\sim \Psi^{3/2}$. It can be readily shown that this requirement leaves f_{et} well-behaved.

V. SUMMARY

It has been the purpose of the foregoing calculation to demonstrate the existence of shock-like electrostatic solutions to the Vlasov equations and Poisson's equation for cold ions and hot electrons. The more difficult and profound questions associated with these solutions — such as their stability and the means by which they might be formed experimentally — have not been touched. Any virtue which the solutions may have probably lies in their tractability and in the fact that they require no ad hoc or unmanageable dissipative mechanisms to bring the plasma from its upstream state to its downstream state.

The essential features of the solutions may be summarized as follows:

(1) There is a downstream population of negative energy electrons, with number density

$$
\begin{aligned}
n_{et}(-\infty) &= n_o \, \Psi \, (\alpha_o) \\
&= n_o \left(1 - \frac{2}{M^2} \, \frac{e\varphi_o}{KT_e}\right)^{-\frac{1}{2}} \\
&\quad - n_o \left(1 - \operatorname{erf} \sqrt{\frac{e\varphi_o}{KT_e}}\right) e^{\,e\varphi_o/KT_e} \quad .
\end{aligned}
$$

(2) The amplitude φ_o may have any value up to $\frac{e\varphi_o}{KT_e} = \Psi_o < \delta^{-2}$, or $\varphi_o < \frac{KT_e}{e} \frac{M^2}{2}$, where M is the Mach number in units of the upstream ion acoustic speed.

(3) There is no limitation on the Mach number from above, which contrasts sharply with the "laminar" solutions based on the moment equations, which always do have such an amplitude limitation at relatively low values of the Mach number.

The second of the above restrictions could perhaps be relaxed by allowing for trapped ions, but this has not been attempted.

It should be noted that the specific entropy current, $\vec{J}_s \equiv \Sigma \int \vec{v} f \ln f \, d\vec{v}$, is the same upstream as downstream, and there is no production of specific entropy in the sense of classical gas dynamics.

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FIGURE CAPTIONS

FIGURE 1a. General shape of the time-independent electrostatic potential for the shock. We do not specify the detailed shape of $\varphi(x)$,

FIGURE 1b. Trajectory in the ion x, v phase plane. The ions are slowed down, but not turned around, by $\varphi(x)$.

FIGURE 1c. Trajectories in the electron x, v phase plane. Negative energy, or "trapped", electrons, are confined to the shaded area. The separatrix is $v = \pm\ (2e\varphi/m_e)^{\frac{1}{2}}$.

FIGURE 2a. The electron distribution as a function of electron energy. The free electron distribution f_{ef} is determined at $x = +\infty$, but the trapped distribution f_{et} is not. We do not specify f_{et} in detail. f_e may or may not be continuous at $\Sigma = 0$.

FIGURE 2b. The trapped electron charge density as a function of scalar potential. $n_{et}(e\varphi)$ must obey Equations (20) of the text, but is not completely determined by them.

FIGURE 3a. A "potential" $V(\Psi)$ which leads to periodic potential waves with finite periodicity. The amplitude of the waves is determined by the intersections of $V = \Pi$ with $V(\Psi)$. This $V(\Psi)$ will not lead to shock-like solutions.

FIGURE 3b. The required form $V(\Psi)$ must have if Ψ is to go monotonically from $\Psi = 0$ to $\Psi = \Psi_o > 0$ as ξ goes from $+\infty$ to $-\infty$. It is important that both $V(0)$ and $V(\Psi_o)$ be local maxima, and be equal. Ψ_o can be any number less than δ^{-2}.

FIGURE 4. Plot of $U(\Psi)$ for different values of δ^2. Note that $U(\Psi)$, $U'(\Psi)$ and $U''(\Psi)$ are always > 0 for $0 < \Psi < \delta^{-2}$.

FIGURE 5. Drawing of a possible $A(\Psi)$ for a typical $U(\Psi)$. $A(\Psi)$ can be any function which is $> U(\Psi)$ in $0 < \Psi < \Psi_o < \delta^{-2}$, and which has $A'(\Psi)$ and $A''(\Psi)$ positive in $0 < \Psi < \Psi_o$. $A(\Psi)$ must have the same value and slope as $U(\Psi)$ at $\Psi = \Psi_o$, and $U(\Psi) - A(\Psi)$ must go as a negative constant times Ψ^2 near $\Psi = 0$; $A(\Psi)$ is otherwise arbitrary.

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FIG. (1a)

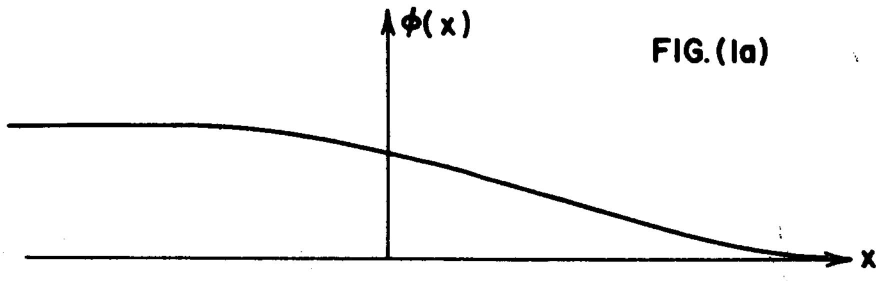

FIG. (1b)

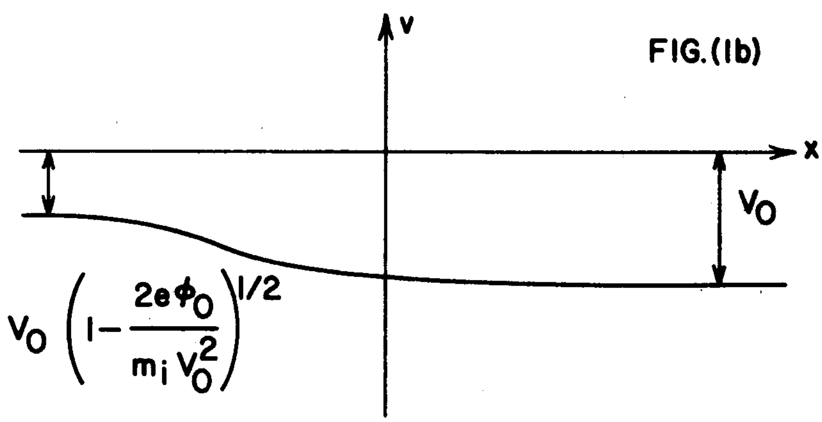

FIG. (1c)

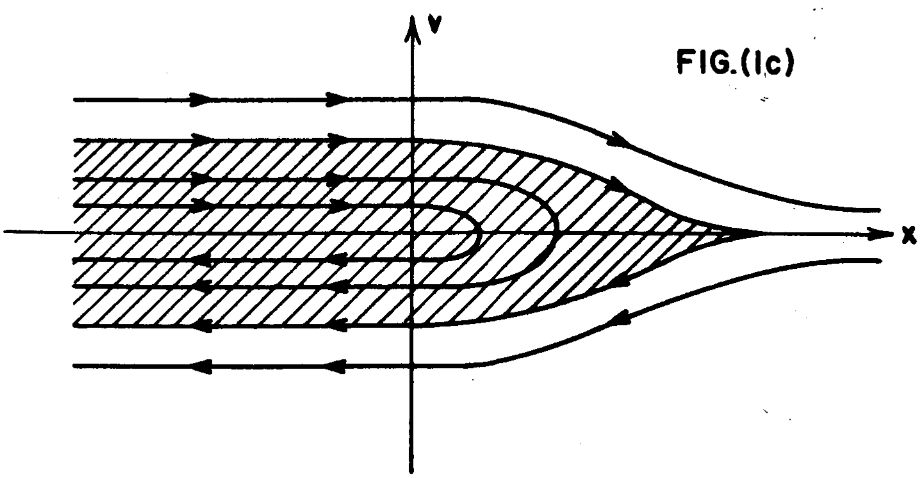

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FIG.(2a)

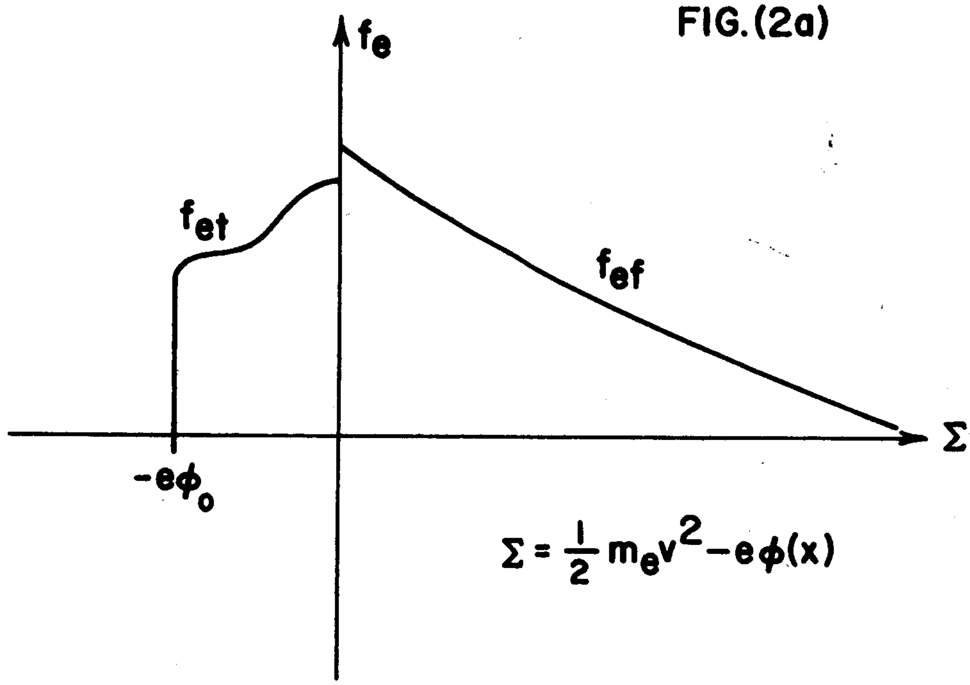
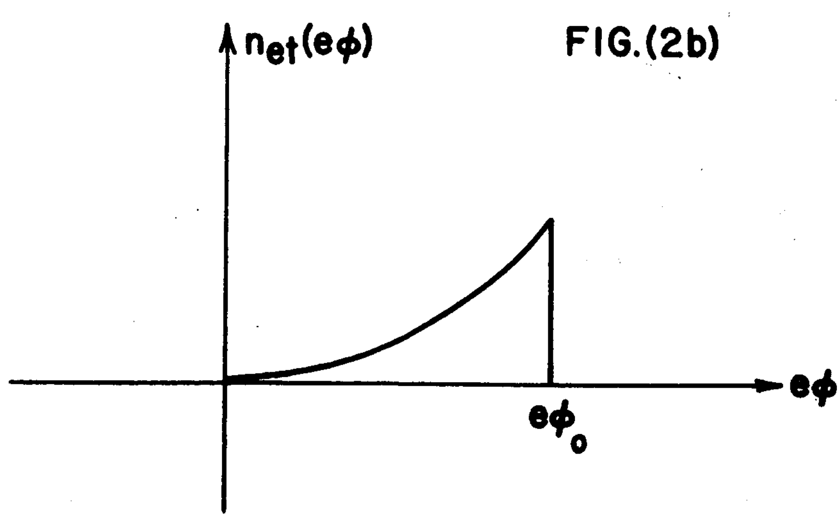


FIG.(2b)



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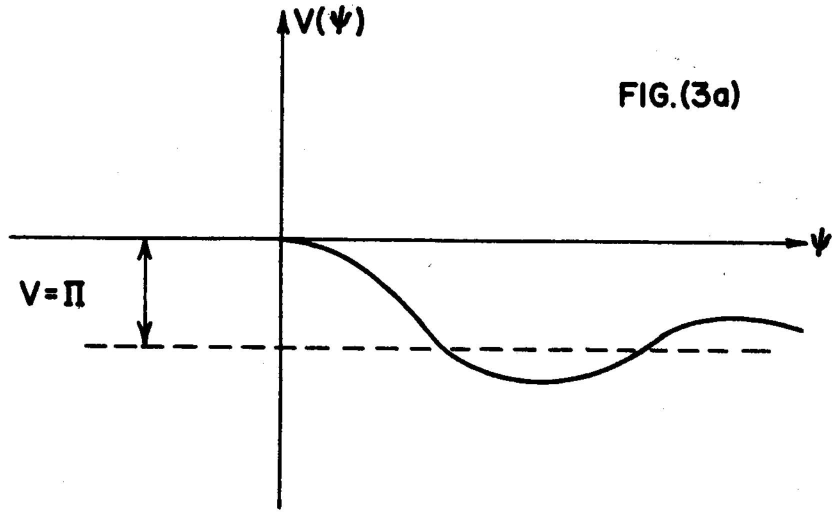

FIG.(3a)

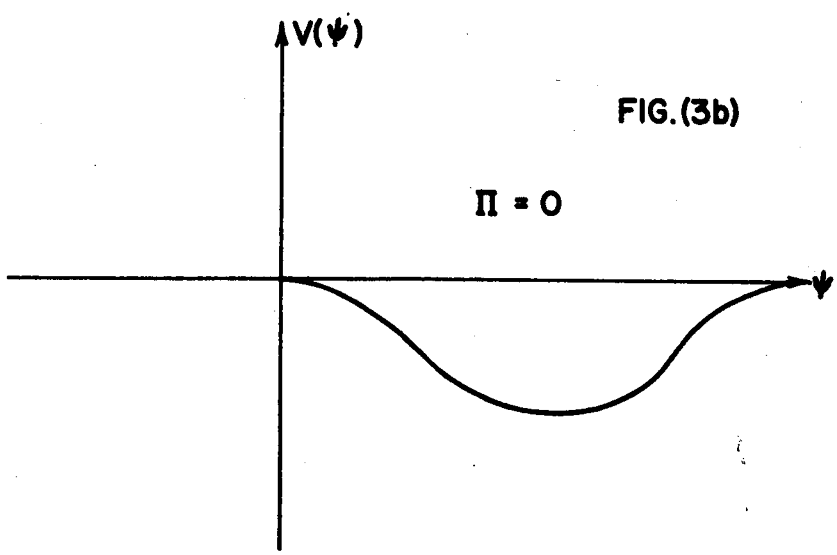

FIG.(3b)

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FIG. (4)

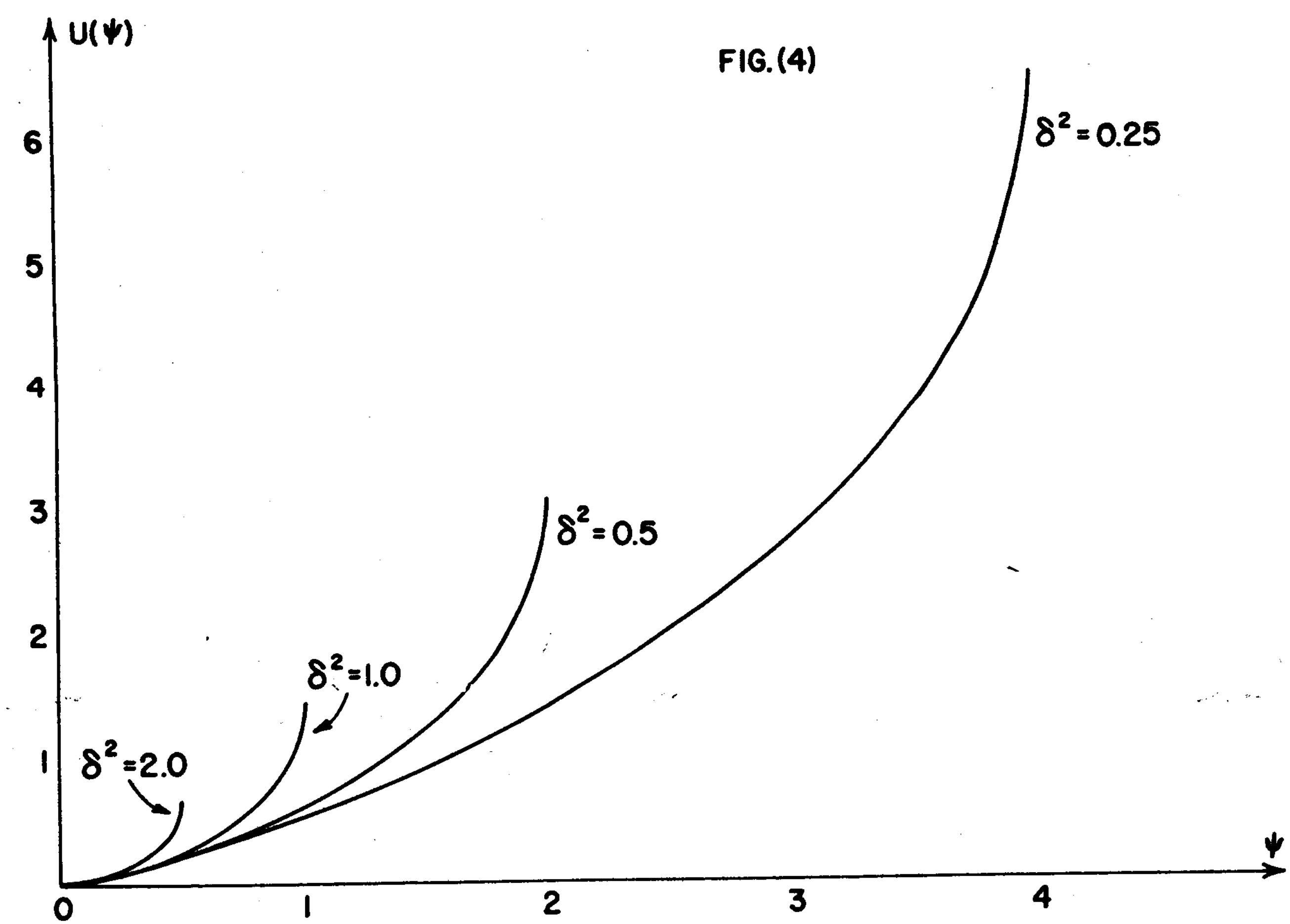

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FIG. (5)